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# Longitudinal Coupling Impedance for a Cavity and Beam Pipe

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## I. Introduction

The solution for the electromagnetic fields in a cavity - beam pipe combination driven by a periodic current source is complex, even for the simplest geometries. Keil and Zotter[1] have obtained the result for the longitudinal coupling impedance for a beam pipe of circular cross section and large circumference connected to a cylindrical cavity. They match field solutions within the beam, between the beam and the beam pipe walls, and in the cavity outside the beam pipe radius, and obtain the result for the coupling impedance as a slowly convergent infinite series.

In this paper we explore the possibility of matching field solutions in two different axial regions: The beam pipe (of circular cross section) and a cavity of general (azimuthally symmetric) shape, in the hopes that the result can be expressed as a sum over just a few cavity modes. In this way it may be possible to evaluate the coupling impedance of an obstacle of general shape by using existing numerical programs such as SUPERFISH[2].

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[1]E. Keil and B. Zotter, Particle Accelerators 3, 11 (1972); see also Warnock, Bart and Fenster, Particle Accelerators 12, 179 (1982).

[2]K. Halbach and R.F. Holsinger, Particle Accelerators 7, 213 (1976).

## II. Analysis of the Fields

We consider a beam pipe of cross sectional radius b and circumferential length $2\pi R$ in which an azimuthally symmetric cavity-like obstacle with dimensions small compared to R is located, as shown in Figure 1.

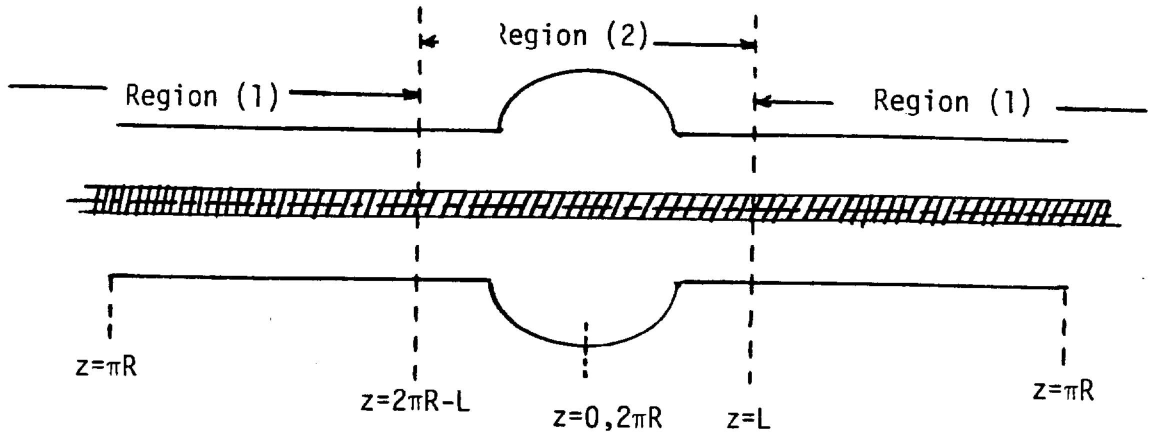


Figure 1.

The longitudinal coupling impedance is defined as[3]

$$Z_L(\omega) = -\frac{V_o}{I_o} \tag{2.1}$$

where the driving current is

$$J_z(r,z,t) = \begin{cases} (I_o/\pi a^2)\, e^{i\omega z/v - i\omega t}, & r<a \\ 0 & r>a \end{cases} \tag{2.2}$$

and the resulting "voltage" is defined by

[3] See, for example, A.W. Chao, 1982 Summer School Lectures, SLAC, p. 396.

$$V_o\, e^{-i\omega t} \equiv \int_o^{2\pi R} dz\, E_z(z)\, e^{-i\omega z/v} \tag{2.3}$$

Here $E_z(z)$ is the z component of the electric field generated by the driving current, averaged over the beam.

The solution for the fields for a lossless beam pipe without an obstacle is well known[4], and readily obtained. Specifically, suppressing the factor $e^{-i\omega t}$,

$$E_z^P = \frac{I_o}{\pi a^2} \frac{1}{i\omega\varepsilon} \begin{Bmatrix} 1 - \sigma a F_1(\sigma a) I_o(\sigma r) \\ \\ \sigma a F_o(\sigma r) I_1(\sigma a) \end{Bmatrix} e^{i\omega z/v} \tag{2.4}$$

$$E_r^P = \frac{I_o}{\pi a^2} \frac{1}{i\omega\varepsilon} \left(\frac{\omega}{\sigma v}\right) \begin{Bmatrix} \sigma a F_1(\sigma a) I_1(\sigma r) \\ \\ \sigma a F_1(\sigma r) I_1(\sigma a) \end{Bmatrix} i\, e^{i\omega z/v} \tag{2.5}$$

$$H_\phi^P = \frac{I_o}{\pi a^2} \frac{1}{\sigma} \begin{Bmatrix} \sigma a F_1(\sigma a) I_1(\sigma r) \\ \\ \sigma a F_1(\sigma r) I_1(\sigma a) \end{Bmatrix} e^{i\omega z/v} \tag{2.6}$$

where

$$F_o(x) = K_o(x) - \frac{K_o(\sigma b)}{I_o(\sigma b)} I_o(x) \tag{2.7}$$

$$F_1(x) = K_1(x) + \frac{K_o(\sigma b)}{I_o(\sigma b)} I_1(x) = -F_o'(x) \tag{2.8}$$

$$\sigma = \frac{\omega}{v\gamma} \tag{2.9}$$

$$\frac{\omega}{v} = \frac{n}{R} \tag{2.10}$$

[4]See, for example, Nielsen, Sessler and Symon, Proc. of the Int'l. Conf. on High Energy Accelerators, Geneva, 1959, p. 239.

and where the upper and lower entries in { } correspond to r<a and r>a.

It is also convenient to expand Eqs. (2.4) - (2.6) in the complete set of radial functions $J_o(p_\ell r/b)$ where $p_\ell$ is the $\ell^{th}$ zero of $J_o(x)$. The result is

$$E_z^P = \frac{I_o}{\pi a^2}\frac{1}{i\omega\epsilon}\; e^{i\omega z/v} \sum_{\ell=1}^{\infty} a_\ell\, J_o\left(\frac{p_\ell r}{b}\right) \tag{2.11}$$

$$E_r^P = \frac{I_o}{\pi a^2}\frac{1}{i\omega\epsilon}\left(\frac{\omega}{\sigma v}\right) i\, e^{i\omega z/v} \sum_{\ell=1}^{\infty} \frac{p_\ell}{\sigma b}\, a_\ell J_1\left(\frac{p_\ell r}{b}\right) \tag{2.12}$$

$$H_\phi^P = \frac{I_o}{\pi a^2}\frac{1}{\sigma}\; e^{i\omega z/v} \sum_{\ell=1}^{\infty} \frac{p_\ell}{\sigma b}\, a_\ell\, J_1\left(\frac{p_\ell r}{b}\right) \tag{2.13}$$

where

$$a_\ell = \frac{J_\ell\left(\frac{p_\ell a}{b}\right)}{\frac{p_\ell a}{2b}\; J_1^2(p_\ell)}\;\frac{\sigma^2\, a^2}{\sigma^2 b^2 + p_\ell^2} \tag{2.14}$$

We shall now write the fields in the actual configuration of Figure 1 as

$$\vec{E} = \vec{E}^P + \vec{E}^C, \quad \vec{H} = \vec{H}^P + \vec{H}^C \tag{2.15}$$

where the superscripts P and C stand for beam pipe and cavity respectively. Since $\vec{E}^P$, $\vec{H}^P$ also correspond to our particular solution, including the driving current, $\vec{E}^C$ and $\vec{H}^C$ will satisfy the usual Maxwell equations without current and charge, namely

$$\nabla \times \vec{E}^C = i\omega\mu\, \vec{H}^C, \qquad \nabla \times \vec{H}^C = -i\omega\varepsilon\, \vec{E}^C \tag{2.16}$$

We now separate our problem into two regions, in which the following boundary conditions apply:

Region (1) Since $\vec{E}^P$ and $\vec{H}^P$ satisfy the boundary conditions on the beam pipe wall, so must $\vec{E}^C$ and $\vec{H}^C$. Thus

$$\vec{E}^C_{tan.} = 0, \qquad \vec{H}^C_{norm.} = 0 \qquad \text{[Region (1) boundary]} \tag{2.17}$$

Region (2) The boundary of Region (2) can extend beyond r=b, and in general $\vec{E}^P$ and $\vec{H}^P$ will not satisfy the correct boundary condition on the cavity walls. Thus we must require

$$\vec{E}^C_{tan.} = -\vec{E}^P_{tan.}, \qquad \vec{H}^C_{norm.} = -\vec{H}^P_{norm.} \qquad \text{[Region (2) boundary]} \tag{2.18}$$

If we are in a frequency region where only one mode can propagate in the beam pipe ($p_1/b<\omega/c<p_2/b$) and if we choose L to be at least several beam pipe diameters, then the cavity fields will coincide with the propagating mode field at z = L, $2\pi R$-L.

We shall evaluate the impedance in Eq. (2.1) separately for even and odd driving currents. Considering the symmetric modes, with a symmetric cavity first, we can replace the z dependent factors in Eqs. (2.4) - (2.6), (2.11) - (2.12) as follows:

$$\left.\begin{aligned} e^{i\omega z/v} &\rightarrow \cos \frac{\omega}{v} (\pi R - z) \\ i\, e^{i\omega z/v} &\rightarrow \sin \frac{\omega}{v} (\pi R - z) \end{aligned}\right\} \tag{2.19}$$

We can now write for $\vec{E}^C$ and $\vec{H}^C$ in Region (1)

$$E_z^C = \frac{A}{i\omega\varepsilon} \cos \alpha_1(\pi R - z) \, J_0(p_1 r/b) \tag{2.20}$$

$$E_r^P = \frac{A}{i\omega\varepsilon} \frac{\alpha_1}{\sigma} \sin \alpha_1(\pi R - z) \, J_1(p_1 r/b) \tag{2.21}$$

$$H_\phi^P = A \frac{1}{\sigma} \cos \alpha_1(\pi R - z) \, J_1(p_1 r/b) \tag{2.22}$$

where

$$\alpha_\ell^{\,2} = (\omega/c)^2 - (p_\ell/b)^2 \tag{2.23}$$

and where A is to be determined. If we choose L to satisfy

$$\alpha_1 \, (\pi R - L) = m\pi \tag{2.24}$$

then $\vec{E}^C$ and $\vec{H}^C$ in (2.20) - (2.22) will satisfy the usual metal wall boundary condition ($\vec{E}_{tan} = \vec{H}_{norm} = 0$) on the interface between Regions (1) and (2) at $z = L$, $2\pi R - L$. In this way the complete cavity problem (Region (2)) is specified by (a) the equations for $\vec{E}^C$ and $\vec{H}^C$ (Eq. (2.16)), (b) the boundary condition, Eq. (2.18), which holds on the outer cavity boundary, and by (c) the boundary condition, Eq. (2.17), which holds at $z = \pm L$ (or at $z = 0$, $z = L$ for a "half cavity"). These represent the standard ingredients for a SUPERFISH[2] calculation without the frequency search/fictitious driving current feature usually needed for finding the cavity eigenmodes.

The output of the SUPERFISH calculation will be the fields $\vec{E}^C$ and $\vec{H}^C$ in Region (2). Because only one mode can propagate in the pipe region, the coefficient A will be determined by matching the cavity fields $E_z^C$ and $H_\phi^C$ to the values $A/i\omega\varepsilon$ and $A/\sigma$ respectively as given in Eqs. (2.20) - (2.22) at $z = L$.

The formulation for the odd driving current is quite similar. In this case Eq. (2.19) is replaced by

$$\left.\begin{array}{lcl} e^{i\omega z/v} & \rightarrow & -i \sin \frac{\omega}{v} (\pi R - z) \\ i\, e^{i\omega z/v} & \rightarrow & i \cos \frac{\omega}{v} (\pi R - z) \end{array}\right\} \quad (2.25)$$

We then obtain for $\vec{E}^C$ and $\vec{H}^C$ in Region (1)

$$E_z^C = \frac{B}{\omega\epsilon} \sin \alpha_1(\pi R - z)\, J_0(p_1 r/b) \quad (2.26)$$

$$E_r^C = \frac{B}{\omega\epsilon} \frac{\alpha_1}{\sigma} \cos \alpha_1(\pi R - z)\, J_1(p_1 r/b) \quad (2.27)$$

$$H_\phi^C = B \frac{i}{\sigma} \sin \alpha_1(\pi R - z)\, J_1(p_1 r/b) \quad (2.28)$$

If we again choose L to obey Eq. (2.24), then $E_z^C$ and $H_\phi^C$ will now vanish at $z = L$, $2\pi R-L$, thus satisfying a "magnetic" wall boundary condition ($\vec{E}_{norm} = \vec{H}_{tan} = 0$) on the interface between Regions (1) and (2). Our cavity problem (Region (2)) then is specified by (a) the equations for $\vec{E}_C$ and $\vec{H}_C$ (Eq. (2.16)), (b) the boundary condition, Eq. (2.18), which holds on the outer cavity boundary, and by (c) the "magnetic" boundary condition

$$\vec{E}^C_{norm.} = 0, \quad \vec{H}^C_{tan.} = 0 \quad (2.29)$$

at $z = 0$ (because the mode is odd in z) and at $z = L$. Once again, these are the standard ingredients for a SUPERFISH calculation without the frequency search. The coefficient B in Eqs. (2.26) - (2.28) will be determined by matching the cavity field $E_r^C$ to the value $B\alpha_1/\omega\epsilon\sigma$ in Eq. (2.27) at $z = L$.

## III. Longitudinal Coupling Impedance

Once the SUPERFISH calculations are completed, the longitudinal coupling impedance can be readily calculated. That portion due to the beam pipe fields is obtained from Eqs. (2.1) - (2.4) by averaging over the beam, and is

$$Z_L^P = \frac{4iR}{\omega\varepsilon a^4} \int_0^a rdr \{ 1 - \sigma a F_1(\sigma a) I_0(\sigma r) \} =$$

$$= \frac{2iR}{\omega\varepsilon a^2} \{ 1 - 2\cdot a F_1(\sigma a) I_1(\sigma a) \} \tag{3.1}$$

For $\sigma b \ll 1$ one can use the series expansions for $K_n(x)$ and $I_n(x)$ for small argument to obtain

$$\frac{Z_L^P}{n} = \frac{iZ_o}{\beta\gamma^2} \left( \tfrac{1}{4} + \ln \frac{b}{a} \right) \tag{3.2}$$

where $Z_o = \sqrt{\mu/\varepsilon} = 377$ ohms is the impedance of free space, in agreement with the well known result for a beam pipe[4].

The contribution of the cavity is obtained by calculating the voltage due to the additional cavity terms:

$$V_o^C = 2i \int_o^L dz \left\langle \frac{E_z^C}{i} \right\rangle_{even} \cos(\omega z/v) - 2i \int_o^L dz \langle E_z^C \rangle_{odd} \sin(\omega z/v) +$$

$$+ \frac{2A}{i\omega\varepsilon} \langle J_o(p_1 r/b) \rangle \int_L^{\pi R} dz \cos \alpha_1(\pi R - z) \cos(\omega z/v) +$$

$$+ \frac{2B}{i\omega\varepsilon} \langle J_o(p_1 r/b) \rangle \int_L^{\pi R} dz \sin \alpha_1(\pi R - z) \sin(\omega z/v) \tag{3.3}$$

where < > stands for the average over the beam. Our solution for $E_z^P$ in Eq. (2.4) makes it clear that $E_z^P$ is imaginary for the current perturbation even in z, and real for the current perturbation odd in z. The same is therefore true for $E_z^C$, and leads to the conclusion that A and B are real, implying an overall imaginary result for $V_o^C$ and $Z_L$.

The last two terms in Eq. (3.3) are readily evaluated, since

$$< J_o(p_1 r/b) > = \frac{J_1(\frac{p_1 a}{b})}{(\frac{p_1 a}{2b})} \tag{3.4}$$

and

$$\int_L^{\pi R} dz \cos \alpha_1(\pi R-z) \cos(\omega z/v) = - \frac{\frac{\omega}{v} \sin(\omega L/v)}{(\omega/v)^2 - \alpha_1^2} \cos(m\pi) \tag{3.5}$$

$$\int_L^{\pi R} dz \sin \alpha_1(\pi R-z) \sin(\omega z/v) = \frac{\alpha_1 \sin(\omega L/v)}{(\omega/v)^2 - \alpha_1^2} \cos(m\pi) \tag{3.6}$$

where we have used Eqs. (2.10) and (2.24) to eliminate all remaining dependence on R. Thus the effect of the cavity with lossless walls is to add an imaginary contribution $Z_L^C = -V_o^C/I_o$ to $Z_L^P$ in Eq. (3.1) or (3.2) which does not depend on R. (The cavity contribution $Z_L^C/n$ will therefore be proportional to 1/R.)

## IV. Two Propagating Pipe Modes

The formulation in Sections II and III will lead to a well defined SUPERFISH problem for the cavity when only a single mode can propagate in the beam pipe. It can also readily be adapted to a frequency below the cut-off frequency of the beam pipe by setting $\vec{E}^C$ and $\vec{H}^C$ to zero in Region (1). However, when two or more modes can propagate in the pipe, the situation is much more complicated.

The following method can be used when two modes can propagate. In this case Eqs. (2.20) and (2.22) are replaced by

$$\left\{ \begin{matrix} E_z^C \\ H_\phi^C \end{matrix} \right\} = \left\{ \begin{matrix} (i\omega\varepsilon)^{-1} \\ \sigma^{-1} \end{matrix} \right\} \left[ A_1 J_{\{^0_1\}}(p_1 r/b)\cos\alpha_1(\pi R-z) + A_2 J_{\{^0_1\}}(p_2 r/b)\cos\alpha_2(\pi R-z) \right] \tag{4.1}$$

It is now not possible to find a value, $z = L$, for which either $\partial H_\phi^C/\partial z = 0$ (metallic boundary) or $H_\phi^C = 0$ ("magnetic" boundary). But if we choose L such that

$$\frac{1}{H_\phi^C} \frac{\partial H_\phi^C}{\partial n} \Bigg|_{z=L} = -\alpha_1 \tan\alpha_1(\pi R-L) = -\alpha_2 \tan\alpha_2(\pi R-L) \equiv -\lambda \tag{4.2}$$

for <u>each</u> of the two terms in Eq. (4.1), and therefore for any linear combination of the two terms, then we can redefine the boundary conditions for $H_\phi^C$ in Region (2) at $z = L$ as

$$\left[ \frac{\partial H_\phi^C}{\partial n} + \lambda H_\phi^C \right]_{z=L} = 0 \tag{4.3}$$

Thus we have a linear combination of the Dirichlet and Neumann boundary condition at $z = L$, and this is sufficient to define the SUPERFISH problem for the cavity. Clearly L and $\lambda$ must first be obtained by solving Eq. (4.2) numerically before performing the SUPERFISH calculation with the boundary condition in Eq. (4.3).

Once the SUPERFISH solution is obtained, $A_1$ and $A_2$ can be found by matching the r dependence at $z = L$. One then performs an analogous calculation for the field solution odd in z and eventually obtains two additional terms in Eq. (3.3) involving $A_2$ and $B_2$. The coupling impedance due to the cavity, $Z_L^C = -V_0^C/I_0$, is obtained as before.

## V. Summary

The calculation of the longitudinal coupling impedance for a cavity of general (azimuthally symmetric shape) attached to a long beam pipe of circular cross section has been formulated as a boundary value problem in two separate longitudinal regions. Specifically, we write the well known solution for the fields of a sinusoidally varying driving current in a uniform beam pipe and express the actual field as the sum of the known beam pipe field and a supplementary field caused by the cavity. The equations for this supplementary field are then written in two regions, one involving just the beam pipe, and the other involving the cavity and just enough of the beam pipe for the evanescent modes to decay. By careful selection of the location of the interface between the two regions, the equations for the supplementary field in the cavity region become Maxwell's equations without current or charge sources, and with well specified (inhomogeneous) boundary conditions on the walls of the cavity. As a result, numerical programs such as SUPERFISH can be directly used to obtain the supplementary field and the corresponding contribution of the cavity to the coupling impedance.

## VI. Acknowledgement

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